JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 24, 182–190 (1968)

Fuzzy Topological Spaces*

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1. INTRODUCTION

The concept of a fuzzy set, which was introduced in [1], provides a natural framework for generalizing many of the concepts of general topology to what might be called fuzzy topological spaces. In the interest of brevity, we shall confine our attention in this note to the more basic concepts such as open set, closed set, neighborhood, interior set, continuity and compactness, following closely the definitions, theorems and proofs given in Kelly [2]. Our notation and terminology for fuzzy sets follow that of Zadeh [1].

2. FUZZY TOPOLOGICAL SPACES

Let $X = \{x\}$ be a space of points. Informally, a fuzzy set A in X is a "class" with fuzzy boundaries, e.g., the "class" of real numbers which are much larger than, say, 10. Such a class is characterized by a membership (characteristic) function which associates with each x its "grade of membership," $\mu_A(x)$, in A. We shall assume that μ_A is a function from X to $[0, 1]$. Many of our definitions and results, however, can readily be extended to the case where μ_A is a function from X to a lattice, as in the work of Goguen [3].

We begin with several preliminary definitions.

DEFINITION 2.1. Let A and B be fuzzy sets in a space $X = \{x\}$, with the grades of membership of x in A and B denoted by $\mu_A(x)$ and $\mu_B(x)$, respectively. Then

$$\begin{aligned}
A = B &\Leftrightarrow \mu_A(x) = \mu_B(x) && \text{for all} \quad x \in X.\\
A \subset B &\Leftrightarrow \mu_A(x) \leqslant \mu_B(x) && \text{for all} \quad x \in X.\\
C = A \cup B &\Leftrightarrow \mu_C(x) = \mathrm{Max}[\mu_A(x), \mu_B(x)] && \text{for all} \quad x \in X.\\
D = A \cap B &\Leftrightarrow \mu_D(x) = \mathrm{Min}[\mu_A(x), \mu_B(x)] && \text{for all} \quad x \in X.\\
E = A' &\Leftrightarrow \mu_E(x) = 1 - \mu_A(x) && \text{for all} \quad x \in X.
\end{aligned}$$

* This work was supported in part by the National Science Foundation under Grant GP-2413 to the Electronics Research Laboratory, University of California, Berkeley, California.

More generally, for a family of fuzzy sets, $A = \{A_i ; i \in I\}$, the union, $C = \bigcup_I A_i$, and the intersection, $D = \bigcap_I A_i$, are defined by

$$\mu_C(x) = \sup_I\{\mu_{A_i}(x)\}, \qquad x \in X$$
$$\mu_D(x) = \inf_I\{\mu_{A_i}(x)\}, \qquad x \in X.$$

The symbol Φ will be used to denote an empty fuzzy set ($\mu_\Phi(x) = 0$ for all x in X). For X, we have by definition $\mu_X(x) = 1$ for all x in X.

We are now ready to define a fuzzy topological space.

DEFINITION 2.2. A fuzzy topology is a family T of fuzzy sets in X which satisfies the following conditions:

(a) Φ, $X \in T$,

(b) If A, $B \in T$, then $A \cap B \in T$,

(c) If $A_i \in T$ for each $i \in I$, then $\bigcup_I A_i \in T$.

T is called a fuzzy topology for X, and the pair (X, T) is a fuzzy topological space, or fts for short. Every member of T is called a T-open fuzzy set. A fuzzy set is T-closed if and only if its complement is T-open. In the sequel, when no confusion is likely to arise, we shall call a T-open (T-closed) fuzzy set simply an open (closed) set. As (ordinary) topologies, the indiscrete fuzzy topology contains only Φ and X, while the discrete fuzzy topology contains all fuzzy sets. A fuzzy topology U is said to be coarser than a fuzzy topology T if and only if $U \subset T$.

DEFINITION 2.3. A fuzzy set U in a fts (X, T) is a neighborhood, or nbhd for short, of a fuzzy set A if and only if there exists an open fuzzy set 0 such that $A \subset 0 \subset U$.

The above definition differs somewhat from the ordinary one in that we consider here a nbhd of a fuzzy set instead of a nbhd of a point.

THEOREM 2.1. *A fuzzy set A is open iff for each fuzzy set B contained in A, A is a* nbdh *of B.*

PROOF. ($\Rightarrow$) obvious.

($\Leftarrow$) Since $A \subset A$, there exists an open fuzzy set 0 such that $A \subset 0 \subset A$. Hence, $A = 0$ and A is open. Q.E.D.

The nbhd system of a fuzzy set is the family of all nbhd's of the fuzzy set.

THEOREM 2.2. *If U is the* nbhd *system of a fuzzy set, then finite intersections of members of U belong to U, and each fuzzy set which contains a member of U belongs to U.*

PROOF. If R and S are nbhd's of a fuzzy set A, there are open nbhd's R_0 and S_0 contained in R and S, respectively. Then $R \cap S$ contains the open nbhd $R_0 \cap S_0$ and is hence a nbhd of A. Thus the intersection of two (and hence of any finite number of) members of U is a member of U. Hence, if a fuzzy set R contains a nbhd of A it contains an open nbhd of A and consequently is itself a nbhd. Q.E.D.

DEFINITION 2.4. Let A and B be fuzzy sets in a fts (X, T), and let $A \supset B$. Then B is called an interior fuzzy set of A iff A is a nbhd of B. The union of all interior fuzzy sets of A is called the interior of A and is denoted by A^0.

THEOREM 2.3. *Let A be a fuzzy set in a* fts (X, T). *Then A^0 is open and is the largest open fuzzy set contained in A. The fuzzy set A is open iff $A = A^0$.*

PROOF. By Definition 2.4, clearly, A^0 is itself an interior fuzzy set of A. Hence there exists an open fuzzy set 0 such that $A^0 \subset 0 \subset A$. But 0 is an interior fuzzy set of A, hence $0 \subset A^0$. Hence $A^0 = 0$. Thus, A^0 is open and is the largest open fuzzy set contained in A. If A is open, then $A \subset A^0$, for A is an interior fuzzy set of A. Hence, $A = A^0$. The converse is obviously true. Q.E.D.

3. SEQUENCES OF FUZZY SETS

DEFINITION 3.1. A sequence of fuzzy sets, say $\{A_n, n = 1, 2,...\}$, is eventually contained in a fuzzy set A iff there is an integer m such that, if $n \geqslant m$, then $A_n \subset A$. The sequence is frequently contained in A iff for each integer m there is an integer n such that $n \geqslant m$ and $A_n \subset A$. If the sequence is in a fts (X, T), then we say that the sequence converges to a fuzzy set A iff it is eventually contained in each nbdh of A.

DEFINITION 3.2. Let N be a map from the set of non-negative integers to the set of non-negative integers. Then the sequence $\{B_i, i = 1, 2,...\}$ is a subsequence of a sequence $\{A_n, n = 1, 2,...\}$ iff there is a map N such that $B_i = A_{N(i)}$ and for each integer m there is an integer n such that $N(i) \geqslant m$ whenever $i \geqslant n$.

DEFINITION 3.3. A fuzzy set A in a fts (X, T) is a cluster fuzzy set of a sequence of fuzzy sets iff the sequence is frequently contained in every nbhd of A.

THEOREM 3.1. *If the* nbhd *system of each fuzzy set in a* fts (X, T) *is countable, then;*

(a) *A fuzzy set A is open iff each sequence of fuzzy sets, $\{A_n, n = 1, 2,...\}$, which converges to a fuzzy set B contained in A is eventually contained in A.*

(b) *If A is a cluster fuzzy set of a sequence $\{A_n, n = 1, 2,...\}$ of fuzzy sets, then there is a subsequence of the sequence converging to A.*

PROOF. (a) ($\Rightarrow$) Since A is open, A is a nbhd of B. Hence, $\{A_n, n = 1,...\}$, is eventually contained in A.

($\Leftarrow$) For each $B \subset A$, let $U_1,..., U_n,...$ be the nbhd system of B. Let $V_n = \bigcap_1^n \{U_i\}$. Then $V_1,..., V_n,...$ is a sequence which is eventually contained in each nbhd of B, i.e., $V_1,..., V_n,...$ converges to B. Hence, there is an m such that for $n \geqslant m$, $V_n \subset A$. The V_n are nbhds of B. Therefore, by Theorem 2.1, A is open.

(b) Let $R_1,..., R_n,...$ be the nbdh system of A. Let $S_n = \bigcup_1^n \{R_i\}$. Then $S_1,..., S_n,...$ is a sequence such that $S_{n+1} \subset S_n$ for each n. For every non-negative integer i, choose $N(i)$ such that $N(i) \geqslant i$ and $A_{N(i)} \subset S_i$. Then surely $\{A_{N(i)}, i = 1, 2,...\}$ is a subsequence of the sequence $\{A_n, n = 1, 2,...\}$. Clearly this subsequence converges to A. Q.E.D.

4. F-CONTINUOUS FUNCTIONS

In this section, we generalize the notion of continuity to what we call F-continuous functions. As a preliminary, we shall establish several properties of fuzzy sets induced by mappings.

DEFINITION 4.1. *Let f be a function from X to Y. Let B be a fuzzy set in Y with membership function $\mu_B(y)$. Then the inverse of B, written as $f^{-1}[B]$, is a fuzzy set in X whose membership function is defined by*

$$\mu_{f^{-1}[B]}(x) = \mu_B(f(x)) \qquad \text{for all} \quad x \quad \text{in} \quad X.$$

Conversely, let A be a fuzzy set in X with membership function $\mu_A(x)$. The image of A, written as $f[A]$, is a fuzzy set in Y whose membership function is given by

$$\begin{aligned} \mu_{f[A]}(y) &= \sup_{z \in f^{-1}[y]} \{\mu_A(z)\} \qquad \text{if} \quad f^{-1}[y] \quad \text{is not empty,} \\ &= 0 \qquad\qquad \text{otherwise,} \end{aligned}$$

for all y in Y, where $f^{-1}[y] = \{x \mid f(x) = y\}$

THEOREM 4.1. *Let f be a function from X to Y. Then:*

(a) $f^{-1}[B'] = \{f^{-1}[B]\}'$ *for any fuzzy set* B *in* Y.

(b) $f[A'] \supset \{f[A]\}'$ *for any fuzzy set* A *in* X.

(c) $B_1 \subset B_2 \Rightarrow f^{-1}[B_1] \subset f^{-1}[B_2]$, *where* B_1, B_2 *are fuzzy sets in* Y.

(d) $A_1 \subset A_2 \Rightarrow f[A_1] \subset f[A_2]$, *where* A_1 *and* A_2 *are fuzzy sets in* X.

(e) $B \supset f[f^{-1}[B]]$ *for any fuzzy set* B *in* Y.

(f) $A \subset f^{-1}[f[A]]$ *for any fuzzy set* A *in* X.

(g) *Let* f *be a function from* X *to* Y *and* g *be a function from* Y *to* Z. *Then* $(g \circ f)^{-1}[C] = f^{-1}[g^{-1}[C]]$ *for any fuzzy set* C *in* Z, *where* $g \circ f$ *is the composition of* g *and* f.

PROOF. (a) For all x in X,

$$\mu_{f^{-1}[B']}(x) = \mu_{B'}[f(x)] = 1 - \mu_B[f(x)] = 1 - \mu_{f^{-1}[B]}(x)$$

$$= \mu_{\{f^{-1}[B]\}'}(x).$$

(b) For each $y \in Y$, if $f^{-1}[y]$ is not empty, then

$$\mu_{f[A']}(y) = \sup_{z \in f^{-1}[y]} \{\mu_{A'}(z)\} = \sup_{z \in f^{-1}[y]} \{1 - \mu_A(z)\}$$

$$= 1 - \inf_{z \in f^{-1}[y]} \{\mu_A(z)\},$$

and

$$\mu_{\{f[A]\}'}(y) = 1 - \mu_{f[A]}(y) = 1 - \sup_{z \in f^{-1}[y]} \{\mu_A(z)\},$$

therefore

$$\mu_{f[A']}(y) \geqslant \mu_{\{f[A]\}'}(y)$$

(c)
$$\mu_{f^{-1}[B_1]}(x) = \mu_{B_1}[f(x)]$$

and

$$\mu_{f^{-1}[B_2]}(x) = \mu_{B_2}[f(x)] \quad \text{for all} \quad x \quad \text{in} \quad X.$$

since

$$B_1 \subset B_2, \qquad \mu_{f^{-1}[B_1]}(x) \leqslant \mu_{f^{-1}[B_2]}(x)$$

for all $x \in X$. Hence

$$f^{-1}[B_1] \subset f^{-1}[B_2].$$

(d) $\mu_{f[A_1]}(y) = \sup_{z \in f^{-1}[y]} \{\mu_{A_1}(z)\}$ and $\mu_{f[A_2]}(y) = \sup_{z \in f^{-1}[y]} \{\mu_{A_2}(z)\}$.

Since $A_1 \subset A_2$,

$$\mu_{f[A_1]}(y) \leqslant \mu_{f[A_2]}(y) \qquad \text{for all} \qquad y \in Y.$$

Hence

$$f[A_1] \subset f[A_2].$$

(e) If $f^{-1}[y]$ is not empty,

$$\mu_{f[f^{-1}[B]]}(y) = \sup_{z \in f^{-1}[y]} \{\mu_{f^{-1}[B]}(z)\} = \sup_{z \in f^{-1}[y]} \{\mu_B(f(z))\} = \mu_B(y).$$

If $f^{-1}[y]$ is empty

$$\mu_{f[f^{-1}[B]]}(y) = 0.$$

Therefore,

$$\mu_{f[f^{-1}[B]]}(y) \leqslant \mu_B(y) \qquad \text{for all} \qquad y \in Y.$$

(f) $\mu_{f^{-1}[f[A]]}(x) = \mu_{f[A]}[f(x)] = \sup_{z \in f^{-1}[f(x)]} \{\mu_A(z)\} \geqslant \mu_A(x)$ for all $x \in X$.

(g) For all $x \in X$,

$$\mu_{(g_0 f)^{-1}[C]}(x) = \mu_C[g \circ f(x)] = \mu_C[g[f(x)]]$$
$$= \mu_{g^{-1}[C]}[f(x)] = \mu_{f^{-1}[g^{-1}[C]]}(x).$$

Q.E.D

We are now ready to define F-continuity.

DEFINITION 4.2. A function f from a fts (X, T) to a fts (Y, U) is F-continuous iff the inverse of each U-open fuzzy set is T-open.

Clearly, if f is an F-continuous function on X to Y and g is an F-continuous function on Y to Z, then the composition $g \circ f$ is an F-continuous function on X to Z, for $(g \circ f)^{-1}[V] = f^{-1}[g^{-1}[V]]$ for each fuzzy set V in Z, and using the F-continuity of g and f it follows that if V is open so is $(g \circ f)^{-1}[V]$.

THEOREM 4.2. *If X and Y are* tfs's, *and f is a function on X to Y, then the conditions below are related as follows:* (a) *and* (b) *are equivalent;* (c) *and* (d) *are equivalent;* (a) *implies* (c), *and* (d) *implies* (e).

(a) *The function f is F-continuous.*

(b) *The inverse of every closed fuzzy set is closed.*

(c) *For each fuzzy set A in X, the inverse of every* nbhd *of $f[A]$ is a* nbhd *of A.*

(d) *For each fuzzy set A in X and each* nbhd *V of $f[A]$, there is a* nbhd *W of A such that $f[W] \subset V$.*

(e) *For each sequence of fuzzy sets $\{A_n, n = 1, 2,...\}$ in X which converges to a fuzzy set A in X, the sequence $\{f[A_n], n = 1, 2,...\}$ converges to $f[A]$.*

Proof. (a) $\Leftrightarrow$ (b). This is an immediate consequence of the fact that $f^{-1}[B'] = \{f^{-1}[B]\}'$ for every fuzzy set B in Y.

(a) $\Rightarrow$ (c). If f is F-continuous, A is a fuzzy set in X, and V is a nbhd of $f[A]$, then V contains an open nbhd W of $f[A]$. Since $f[A] \subset W \subset V$, $f^{-1}[f[A]] \subset f^{-1}[W] \subset f^{-1}[V]$. But $A \subset f^{-1}[f[A]]$ and $f^{-1}[W]$ is open. Consequently, $f^{-1}[V]$ is a nbhd of A.

(c) $\Rightarrow$ (d). Since $f^{-1}[V]$ is a nbhd of A, we have $f[W] = f[f^{-1}[V]] \subset V$, where $W = f^{-1}[V]$.

(d) $\Rightarrow$ (c). Let V be a nbhd of $f[A]$. Then there is a nbhd W of A such that $f[W] \subset V$. Hence, $f^{-1}[f[W]] \subset f^{-1}[V]$. Furthermore, since $W \subset f^{-1}[f[W]]$, $f^{-1}[V]$ is a nbhd of A.

(d) $\Rightarrow$ (e). If V is a nbhd of $f[A]$, there is a nbhd W of A such that $f[W] \subset V$. Since $\{A_n, n = 1, 2,...\}$ is eventually contained in W, i.e., there is an m such that for $n \geqslant m$, $A_n \subset W$, we have $f[A_n] \subset f[W] \subset V$ for $n \geqslant m$. Therefore $\{f[A_n], n = 1, 2,...\}$ converges to $f[A]$. Q.E.D.

A fuzzy homeomorphism is an F-continuous one-to-one map of a fts X onto a fts Y such that the inverse of the map is also F-continuous. If there exists a fuzzy homeomorphism of one fuzzy space onto another, the two fuzzy spaces are said to be F-homeomorphic and each is a fuzzy homeomorph of the other. Two fts's are topologically F-equivalent iff they are F-homeomorphic.

5. Compact Fuzzy Spaces

We now consider a fuzzy compact space constructed around a fuzzy topology.

Definition 5.1. A family $\mathbf{A}$ of fuzzy sets is a cover of a fuzzy set B iff $B \subset U\{A \mid A \in \mathbf{A}\}$. It is an open cover iff each member of $\mathbf{A}$ is an open fuzzy set. A subcover of $\mathbf{A}$ is a subfamily of $\mathbf{A}$ which is also a cover.

Definition 5.2. A fts (X, T) is compact iff each open cover has a finite subcover.

DEFINITION 5.3. A family $\mathbf{A}$ of fuzzy sets has the finite intersection property iff the intersection of the members of each finite subfamily of $\mathbf{A}$ is nonempty.

THEOREM 5.1. *A* fts *is compact if and only if each family of closed fuzzy sets which has the finite intersection property has a nonempty intersection.*

PROOF. If $\mathbf{A}$ is a family of fuzzy sets in a fts (X, T), then $\mathbf{A}$ is a cover of X iff $\cup \{A \mid A \in \mathbf{A}\} = X$, or iff $\{\cup [A \mid A \in \mathbf{A}]\}' = X' = \Phi$, or iff $\cap \{A' \mid A \in \mathbf{A}\} = \Phi$ by the De Morgan's laws. Hence, the fuzzy space X is compact iff each family of open fuzzy sets in X such that no finite subfamily covers X, fails to be a cover, and this is true iff each family of closed fuzzy sets which possesses the finite intersection property has a nonempty intersection. Q.E.D.

THEOREM 5.2. *Let f be an F-continuous function carrying the compact* fts X *onto the* fts Y. *Then Y is compact.*

PROOF. Let $\mathbf{B}$ be an open cover of Y. Then, since

$$\mu_{\bigcup_{B\in\mathbf{B}} f^{-1}[B]}(x) = \sup_{B\in\mathbf{B}}\{\mu_{f^{-1}[B]}(x)\} = \sup_{B\in\mathbf{B}}\{\mu_B(f(x))\} = 1 \qquad \text{for all} \qquad x \in X,$$

the family of all fuzzy sets of the form $f^{-1}[B]$, for B in $\mathbf{B}$, is an open cover of X which has a finite subcover. However, if f is onto, then it is easily seen that $f[f^{-1}[B]] = B$ for any fuzzy set B in Y. Thus, the family of images of members of the subcover is a finite subfamily of $\mathbf{B}$ which covers Y and consequently Y is compact. Q.E.D.

6. CONCLUDING REMARKS

The results presented in this note indicate that many of the basic concepts in general topology can readily be extended to fuzzy topological spaces. Although the theory of fuzzy sets is still in an embryonic stage, it shows promise of having wide applications.

ACKNOWLEDGMENT

The author wishes to express his appreciation to Professor L.A. Zadeh for his suggestions and encouragement.

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